GRAPH THEORY

Definition

A graph G consists of

1. A finite set of **vertices**
2. A finite set of **edges**
3. A function [the **edge endpoint function**] that maps each edge to a set of one or two vertices [the **end points** of the edge]

Example

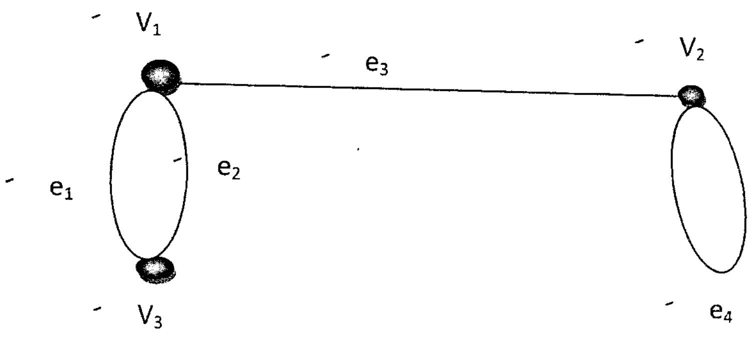
Consider the graph defined as follows

i. ***Its vertices are [v_1, v_2, v_3]***
ii. ***Its edges are [e_1, e_2, e_3, and e_4]***
iii. ***The end points of each edge are given by the table below***

edge	endpoints
e_1	[v_1, v_3]
e_2	[v_1, v_3]
e_3	[v_1, v_2]
e_3	[v_2]

This table describes an edge endpoint function. Draw the graph defined above

Solution



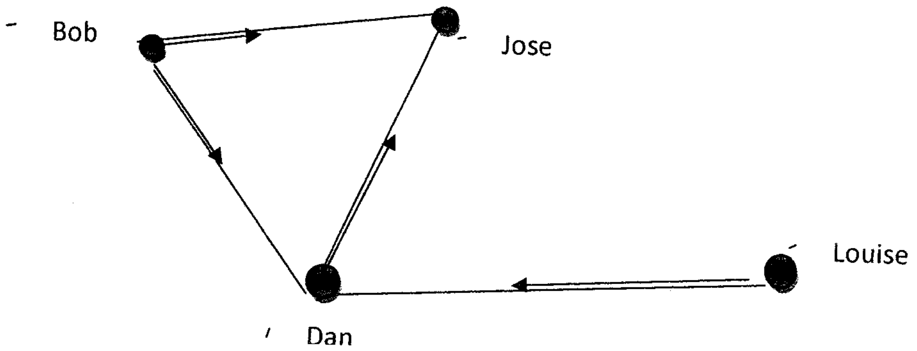
The essential feature of an edge is that it joins or connects its endpoints, its particular shape, curve or segment is not important. The vertices connected by an edge are called **adjacent vertices**. Two edges with a common endpoint are called **adjacent edges**. For instance, e_3 and e_2 in the example above are adjacent edges

Note that vertices v1 and v3 connected by more than one edge. When this occurs the edges are called **parallel.** [This is a different meaning for 'parallel' than that associated with two lines.]Also, edge e4 joins vertex v2 to itself. Such an edge is called a **loop.** Note that in a graph, there must be a vertex at each end of the edge. A vertex which is not an endpoint of any edge in a graph is called an **isolated vertex.**

A graph with no loops and parallel edges is called a **simple graph.**

It is sometimes useful to direction to each edge of a graph. The resulting **digraph** is pictured like the other graphs except that its edges are draw with arrows. The formal definition of the digraph is the same as the definition of graph except that the edge end point function sends each edge to an **ordered pair** of vertices. (That would be indicated in an edge point table by say(v_1,v_2) rather than {v_1,v_2}.)

For instance some group behavior studies investigate the influence one person has on another in some social setting. The directed graph below shows a set of influence relationship.



From the arrows you can see that Bob influences Jose and Dan, and Dan influence only Jose while Louise influences Dan. It can be seen also that Jose does not influence anybody.

Exercise

1. **Draw the picture of the graph defined as follows:**
   i. **Set of vertices {v_1, v_2, v_3, v_4, v_5}**
   ii. **Set of edges {e_1, e_2, e_3, e_4, e_5}**
   iii. **The endpoint function**

edge	endpoints
e_1	{v_1, v_2}
e_2	{v_1, v_4}
e_3	{v_1,v_4}
e_4	{v_5}
e_5	{v_4,v_5}

From the graph you have drawn identify (a) an edge which is a loop (b) two parallel edges (c) an isolated vertex.

2. **Some group behavior studies investigate the influence one person has on another in some social setting. Jose influence Bob, Dan and Joan; and Louise influences Jose, Dan, and Joan. Draw a directed graph to show the influence relationship.**

3. **Draw the picture of directed graph**

MATRIX REPRESENTATION OF GRAPH

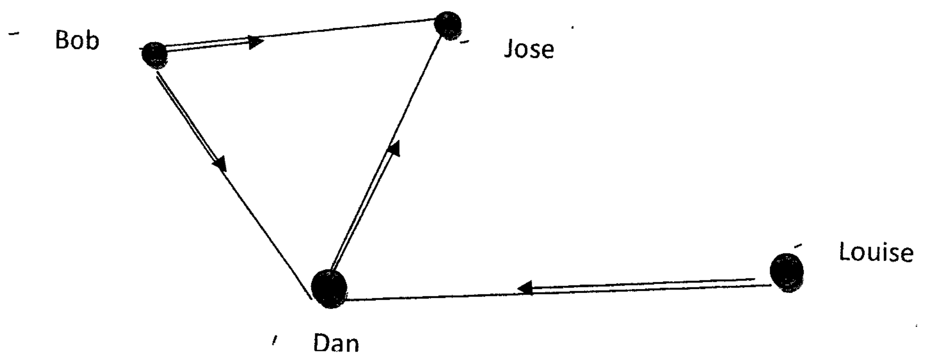
It is natural to want to describe a graph numerically. It is somewhat surprising that this can be done using a matrix.

Definition

The adjacency matrix M for a graph with vertices v_1, v_2,..., v_n is the nxn matrix in which for every i, j, the element in the ith row and jth column is the number of edge from vertex v_i to v_j.

Example

Write the adjacency matrix for the for the directed graph of influence relationship



Solution

Because there are 4 vertices the adjacency matrix will have 4 rows and 4 columns. Label the rows and columns with vertex names. (Bob v_1, Jose v_2, Dan v_3, and Louise v_4). To fill the entry in the ith row and the jth column, just count the number of edges in the vi to vj. For instance, there no edges from v_1 to v_1, so the entry I row 1 and column 1 is 0. There is one edge from v_1 to v_3, so the entry in row 1, column3 is 1. The entire matrix is given below,

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In the matrix for a digraph, each edge appears once in the matrix, so the sum of all elements of the matrix equals the number of edges of the graph. Here that number is 4.

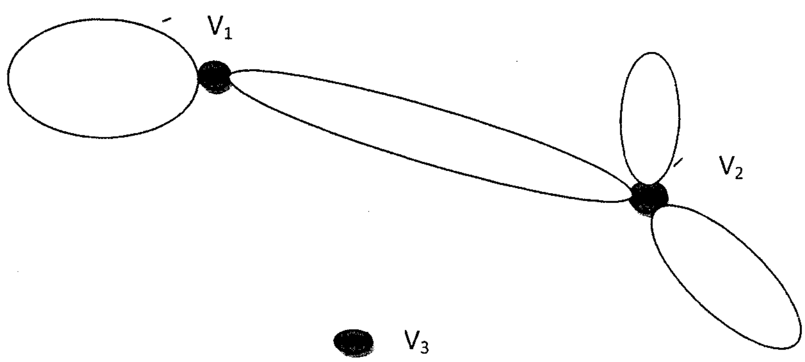
Example

Draw the graph of a graph (not directed) that has the following adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Draw vertices v_1, v_2, and v_3 and connect them by edges as indicated in the matrix. For example, the 2 in the first row and the second column indicate that there are two edges from v_1 to v_2.



Exercise

1. Draw the directed graph with the adjacency matrix given below

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

2. Given the matrix below
   i. Does the adjacency matrix represent a simple graph
   ii. Explain your answer in i above

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

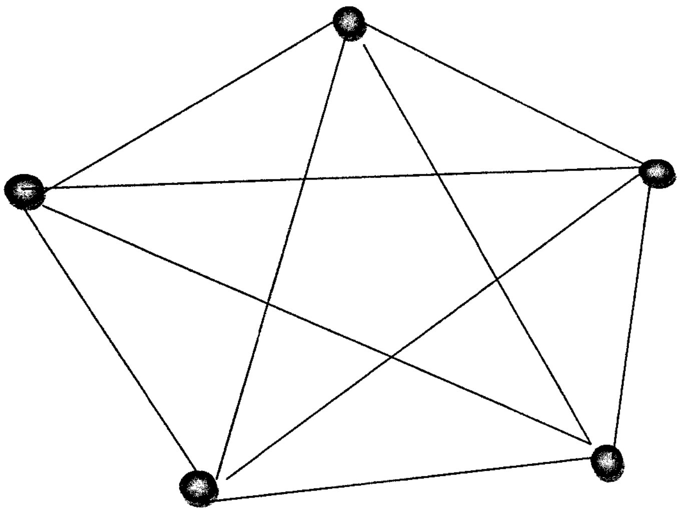
3. Write the adjacency matrix of the following graph defined as follows:
   Vertices :{v1, v2, v3, v4}
   Edges: {e1, e2, e3, e4, e5, e6}
   End point function:

edge	end points
e_1	(v_1, v_2)
e_2	(v_1, v_3)
e_3	(v_3, v_1)
e_4	(v_3, v_3)
e_5	(v_2, v_4)
e_6	(v_3, v_4)

HANDSHAKE PROBLEM

Suppose n people are in a party. If each person shakes hands with every other person, how many hand shakes are required?

This hand shake problem can be represented using graph. Suppose n=5 then the problem can be represented as follows:



Since each person shakes hands with each other person, every pair of vertices is joined by exactly one edge. A graph with this property is called a **complete graph**

You can see that a complete graph ca be pictured the union of a polygon with its diagonal

There number of way to solve the handshake problem. One way is to use combination. Note that there are as many handshakes as there are ways to chose 2 people out of group of n people. This number is

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

For this graph when n is 5 there are 10 hand shakes

Paths and cycles

If we think of vertices in a graph as cities and edges as roads, a **path** is a trip beginning at some city, passing through several cities and terminating at some city.

A **connected graph** is a graph in which given any vertex vi there is a path from v_i to any other vertex v_j

A connected graph consists of one piece while **unconnected graph** is composed of two or more pieces. These pieces are called **sub graphs** of the original graph called the **components**

Note that the definition of a path allows repetition of edges or vertices or both. Subclasses of paths are obtained by prohibiting duplicate vertices and edges or by making v_0 and v_n identical

Definition

Let v and w be vertices in graph G.

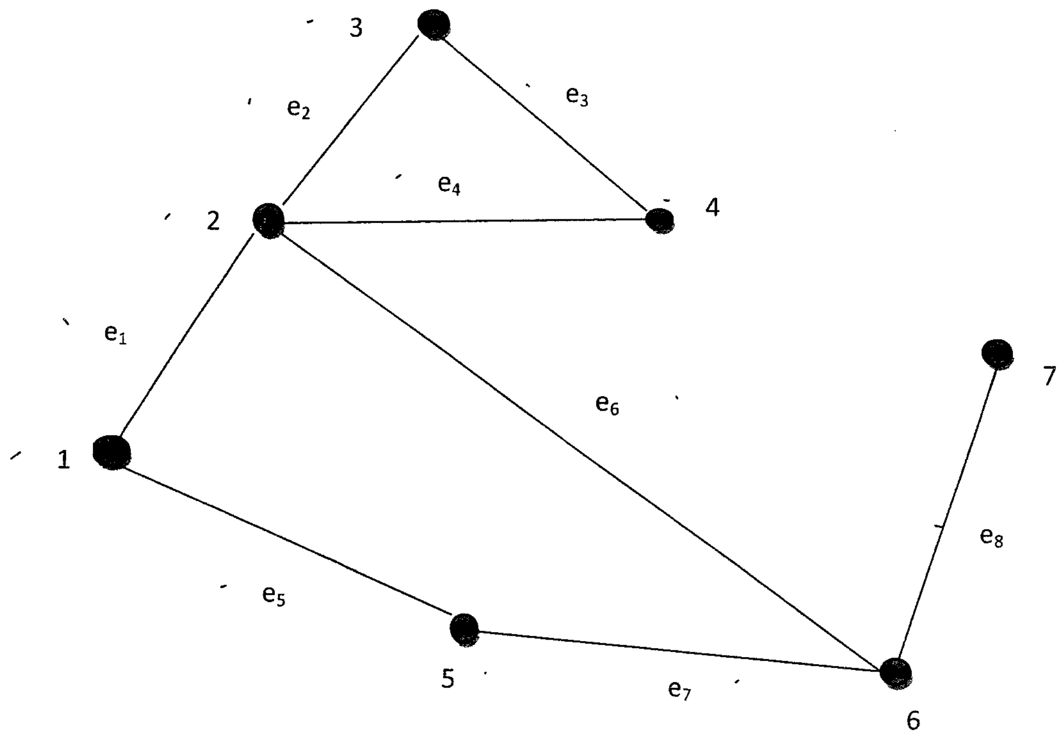
A **simple path** from v to w is with no repeated vertices.

A cycle (or circuit) is a path from v to v with no repeated edges

A **simple cycle** is a cycle from v to v with no repeated vertices

Example

Study the following graph below and answer the questions that follow



(a) Giving reason for your answer state whether the graph is:

(i) Connected graph
(ii) Complete graph
(iii) Simple graph
(iv) digraph

(b) The table below gives some paths from the graph fill (NO or YES)the blanks space in the table to show the type of the given path

path	Simple path?	Cycle?	Simple cycle?
(6, 5, 2, 4, 2, 1)			
(6, 5, 2, 4)			
(2, 6, 5, 2, 4, 3, 2)			
5, 6, 2, 5)			
(7)	X	X	X